On the Solution of a Microstripline with Two Dielectrics

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Abstract —We present the calculation for the capacitance and the effective dielectric constant for a microstripline with two different dielectrics. The solution is based on the exact transformation law provided by two successive Schwarz-Christoffel transformations, which is given in terms of the Jacobi Z_n function. This function can be easily separated into its real and imaginary parts, allowing the exact determination of the curve which separates the two dielectrics in the transformed plane. Once the curve is obtained, the capacitance of the system is calculated numerically by a finite-difference method. We compare our results with data obtained from Wheeler's approximate ellipse solution, as well as with other analytical solutions. We assume an infinitely wide ground plane and TEM-mode propagation.

I. Introduction

FIG. 1(a) SHOWS the geometry of the stripline considered in this paper. Although microstrips have been discussed for some thirty years, no analytical exact solution has been given for the case when two different dielectrics are considered, $\epsilon_1 \neq \epsilon_2$. This is in part due to the fact that the line must be transformed by conformal transformations into the geometry, shown in Fig. 1(b), before the calculation of capacitance can be attempted. The transformation law from the z to the p plane is normally given in terms of elliptic functions and elliptic integrals, thus making the determination of the line that separates the two dielectrics in the plane p difficult.

In the present paper, we review briefly those significant analytical solutions to the microstrip problems, and then proceed to derive the transformation law in a rather simpler way, in terms of Jacobi Z_n functions. From the transformation law, we obtain the function that defines the curve between the dielectrics, and then proceed to solve for the capacitance of the system. We present a comparison of our results with previously published approximate results for the two dielectric case.

II. Previous Theoretical Results

We briefly review some of the pertinent theoretical calculations related to the microstripline.

A. Assadurian and Rimai 1952 [1]

These authors consider the same geometry shown in Fig. 1(a), with the ground plane of infinite extent, assuming a uniform dielectric between and above the metal plates, and consider the case of a wide upper strip ($d/h \gg 1$), so that the problem solved calculated the fringing field at the end of an infinitely wide parallel plate plane condenser. This problem was in fact presented by E. Weber [2] in 1950. This approximate solution for the case of a single dielectric yields impedance values that differ significantly from the correct solution, even in the range of impedances below 50 Ω.

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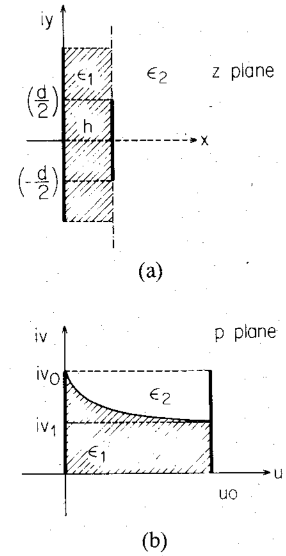

Fig. 1. (a) The microstrip real geometry in z space. (b) The geometry in p plane (assuming an infinite ground plate covered by ϵ_1).

B. Black and Higgins 1958 [3]

These authors consider the geometry shown in Fig. 1(a) for the case of a single dielectric, and solve the problem by exact conformal mapping, considering a ground plane of finite width. Their work results into six equations with six unknowns that must be solved in order to obtain the line parameters. Their procedure is correct, but its application is complicated even for the case of only one dielectric. We will compare later on their finite ground plane, one dielectric solutions, with our solution for an infinite ground plane.

C. Wheeler 1964–65 [4], [5]

This author uses an approximate conformal transformation applied to the geometry of Fig. 1(a). He determines the approximate nature of the curve in the p plane that

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separates the two different dielectrics, and assumes rather empirically that this curve is an ellipse. As we will show later on, the exact curve between the dielectrics is indeed similar to an ellipse in the sense that it intersects the metal plate at $u_0 + iv_1$, with an angle of 90°, and it intersects the metal plate of $0 + iv_0$ with an angle of 0° (as can be inferred from the conformal transformation of the angles in the z plane), but differs substantially from an ellipse in the intermediate region in the p plane. This difference from the ellipse causes substantial differences in the values of calculated impedances, particularly for small d/h ratios and for the case where the two dielectric constants differ strongly. Using an ellipse as the curve between the dielectrics, Wheeler solved the electromagnetic problem and calculated the capacitance of the mixed dielectric media.

D. Schneider 1969 [6], [7]

This author presents the exact conformal transformation for the geometry considered in Fig. 1(a) for the case of one dielectric and when the ground plane is infinite in extent. Schneider's solution is given in terms of the logarithmic derivative of Theta functions; this author did not attempt to consider the case of two different dielectrics.

E. Poh et al. 1981 [8]

In this recent paper, the authors consider the solution for the line capacitance and the characteristic impedance of a microstripline by means of the spectral domain analysis incorporating the edge effect singularity considering two dielectrics and a ground plane of infinite width. We will compare their results with ours later on in the paper.

We have indicated only those references that are of particular interest for the case of two dielectrics. The interested reader is referred to cumulative references for complete bibliographical reviews [9].

II. Solution for the Case of one Dielectric ($\epsilon_1 = \epsilon_2 \equiv \epsilon$)

Fig. 2 indicates the procedure required for transformation of the initial microstrip geometry in the z plane (Fig. 2(a)) into the real axis of the e plane (Fig. 2(b)) by means of a Schwarz-Christoffel transformation of the polygon $Z_1Z_2Z_3Z_4Z_5Z_6$. The real axis of the e plane is then transformed into the rectangular polygon $P_1P_2P_3P_4P_5$ in the p plane (Fig. 2(d)) by means of an inverse Schwarz-Christoffel transformation. The plane indicated in Fig. 2(c) (plane w) is included for illustrative purposes since it arises naturally in the evaluation of some of the integrals.

Once the parameters of the p plane are determined (u_0 and v_0), the capacitance of the microstrip for the simpler one dielectric case can be calculated.

A. The Transformation z to e

By means of a standard Schwarz-Christoffel transformation, the z plane polygon indicated in Fig. 2(a) is transformed into the real axis of the e plane by

$$z = z_0 + A\int_0^e \frac{(e-a')\,de}{\sqrt{e(e-1)(e-b')}} \tag{1}$$

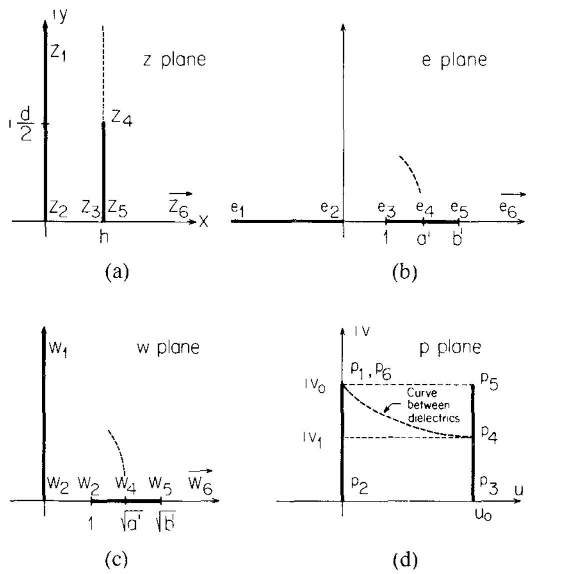

Fig. 2 (a) z plane geometry (b) e plane geometry. (c) w plane geometry. (d) p plane geometry

where z_0, A, a', and b' are constants to be determined by evaluating the correspondence of points in the z and e planes. We introduce a new variable w ($w^2 \equiv e$) so as to obtain integrals directly expressible in the canonical forms of elliptic integrals and elliptic functions [10], [11].

From the correspondence of points $z_3 = h$ and $e_3 = 1$, we obtain

$$h/(-2A) = \frac{1}{\sqrt{b'}}(a'-b')K(m) + \sqrt{b'}\,E(m) \tag{2}$$

where

$$K(m) = \text{the complete elliptic integral of the first kind} \tag{3}$$

$$E(m) = \text{the complete elliptic integral of the second kind} \tag{4}$$

$$m = (1/b'), \qquad \text{the modulus of the integrals.} \tag{5}$$

From the correspondence of points $z_5 = h$ and $e_5 = b'$, we obtain

$$h/(-2A) = \frac{1}{\sqrt{b'}}(a'-b')K(m) + \sqrt{b'}\,E(m) - i\sqrt{b'}\,\{E'(m) - (a'/b')K'(m)\} \tag{6}$$

where

$$K'(m) = K(1-m) \tag{7}$$

$$E'(m) = E(1-m). \tag{8}$$

From the imaginary part of (6), we have

$$a' = b'(E'(m)/K'(m)). \tag{9}$$

The correspondence of points z_4 and e_4 implies

$$\frac{h + i(d/2)}{-2A} = \frac{a'}{\sqrt{b'}}K(m) + \sqrt{b'}\,\{E(m) - K(m)\} - i\left\{\frac{a'}{\sqrt{b'}}F(\phi_1/\alpha_1) - K'(m)\right\} - i\sqrt{b'}\,\{E'(m) - E(\phi_1/\alpha_1)\} \tag{10}$$

where

$F(\phi_1/\alpha_1)$ = the incomplete elliptic integral of the first kind (11)

$E(\phi_1/\alpha_1)$ = the incomplete elliptic integral of the second kind (12)

$$\phi_1 = \sin^{-1}\sqrt{\frac{b'-a'}{b'-1}} \tag{13}$$

$$\alpha_1 = \cos^{-1}(1/\sqrt{b'}). \tag{14}$$

The imaginary part of (10) yields

$$\frac{d/2}{+2A} = \frac{a'}{\sqrt{b'}}\{F(\phi_1/\alpha_1) - K'(m)\} + \sqrt{b'}\{b'E'(m) - a'K'(m)\}. \tag{15}$$

Dividing (15) by (2), and rearranging terms, we obtain

$$\frac{d}{2h} = \frac{-a'F(\phi_1/\alpha_1) + b'E(\phi_1/\alpha_1)}{(a'-b')K(m) + b'E(m)}. \tag{16}$$

If we substitute, in the equation above, the value of a', as given by (9), and if we use the definition of the Jacobi Z_n function

$$Z_n(\phi_1/\alpha_1) \equiv E(\phi_1/\alpha_1) - E(\phi_1)F(\phi_1/\alpha_1)/K(\alpha_1) \tag{17}$$

we can then rewrite (16) as follows:

$$\frac{d}{2h} = (2/\pi)K'(m)Z_n(\phi_1/\alpha_1) = \text{a function of } m. \tag{18}$$

Once m is known from the solution of (18), a' can be determined from (9). The constant A is found in terms of m and b' by substituting (9) in (2), and by using Legendre's [10] relation

$$A = -\{hK'(m)\}/\{\pi\sqrt{b'}\}. \tag{19}$$

B. The Transformation p to e

Applying the normal Schwarz-Christoffel transformation to the polygon $P_1P_2P_3P_4P_5$, defined in Fig. 2(d), in plane p, we obtain the following transformation law from the p to the e plane:

$$p = p_0 + B\int_0^e \frac{de}{\sqrt{e(e-1)(e-b')}}. \tag{20}$$

By establishing the relation between corresponding points in planes e and p, and evaluating (20), we can determine the values of the p plane parameters p_0, u_0, v_1, and v_0 as follows:

$$p_0 = 0 \tag{21}$$

$$u_0/(+2B) = (1/\sqrt{b'})K(m) \tag{22}$$

$$\frac{u_0 + iv_1}{+2B} = (1/\sqrt{b'}\{K(m) + iF(\phi_2/\alpha_2)\} \tag{23}$$

where

$$\phi_2 = \sin^{-1}\sqrt{\frac{b'(a'-1)}{a'(b'-1)}} \tag{24}$$

$$\alpha_2 = \cos^{-1}(1/\sqrt{b'}) \tag{25}$$

so that by separating real and imaginary parts we obtain

$$v_1/(+2B) = (1/\sqrt{b'})F(\phi_2/\alpha_2). \tag{26}$$

The equivalence of points p_5 and e_5 finally yields

$$\frac{u_0 + iv_0}{+2B} = (1/\sqrt{b'})\{K(m) + iK'(m)\} \tag{27}$$

and again by separating real and imaginary parts, we obtain

$$u_0/(+2B) = (1/\sqrt{b'})K(m) \tag{28}$$

$$v_0/(+2B) = (1/\sqrt{b'})K(m). \tag{29}$$

C. The Calculation of the Capacitance for the Single Dielectric Case $\epsilon_1 = \epsilon_2 \equiv \epsilon$

When the microstrip is immersed in a media with uniform permitivity ϵ, the capacitance per unit length of line is easily calculated in the geometry of the p plane. Because of the nature of the solution of the Laplace equation, the electrostatic potential in the region between the metal plates of the p plane is identical to the solution for the potential in an infinite parallel plate condenser. Thus, C_1 (the capacitance per unit length for the one dielectric case) is simply given by

$$C_1 = 2(v_0/u_0) = 2\epsilon\frac{K'(m)}{K(m)} \tag{30}$$

where m is determined from (18) as a function of the parameter of the line in z space: $(h/2d)$; the factor 2 takes into account the fact that we considered half the line only. In the case of a uniform dielectric, the wave propagation mode is TEM, and we can easily determine the line impedance. The velocity of propagation of the wave V is

$$V = \frac{1}{\sqrt{\epsilon\mu}} \equiv \frac{1}{\sqrt{L_1C_1}} \tag{31}$$

where μ is the permeability of the media, and L_1 is the inductance per unit length of line. We determine the value of L_1 from (31), and we substitute its value in the expression for the characteristic impedance of the line Z_1

$$Z_1 = \sqrt{L_1/C_1} = \frac{\sqrt{\mu\epsilon}}{C_1} \tag{32}$$

and finally, using (30), we obtain

$$Z_1 = \left(\sqrt{\frac{\mu}{\epsilon}}\right)\left(\frac{K(m)}{K'(m)}\right). \tag{33}$$

In Fig. 3, we show values of impedance calculated according to our results when the line is in free space ($\epsilon = \epsilon_0$ and $\mu = \mu_0$) for different values of (d/h). Also shown are results obtained by Assadurian and Rimai, Black and Higgins, Wheeler, and Schneider. Table I shows the comparison between numerical results. This comparison between the different results will be discussed later on in the paper.

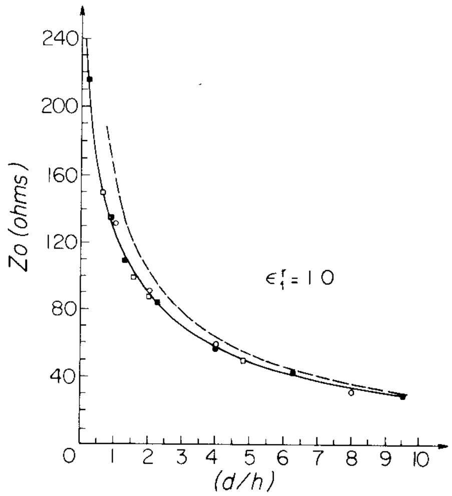

Fig. 3. One dielectric (air) case: Z_0 versus (d/h) Continuous curve: Our results; empty squares: Wheeler; dashed curve: Assadurian and Rimai; filled squares: Schneider; filled circles: Black and Higgins (ratio of top to bottom conductor widths = 4); empty circles: Black and Higgins (ratio of top to bottom conductors widths = 8)

TABLE I

Z_0 (Ω)	d/h Wheeler	d/h Schneider	d/h Black and Higgins	d/h Our Results
210	0.146, (+ 0.14)	0.145, (- 0.14)		0.1458
150	0.666, (+ 0.03)	0.662, (- 0.16)		0.6631
126.958		0.9927, (0.0)		0.9927
100	1.605, (- 0.90)	1.613, (- 0.10)		1.6150
88.132			2.000, (- 1.96)	2.040
56.338			4.000, (- 4.50)	4.189
50	4.840, (- 1.22)	4.786, (- 0.63)		4.9068
30	9.519, (- 0.90)	9.553, (- 0.56)		9.6062
20	15.592, (- 0.20)	15.489, (- 0.86)		15.6230

IV. Solution for the Two Dielectric Case $\epsilon_1 \neq \epsilon_2$

When the dielectric between the metal plates (see Fig. 1(a), z plane geometry) differs from the dielectric above the top metal plate, the problem becomes quite complicated. The waves no longer propagate in the simpler TEM modes; the capacitance cannot be calculated simply, even in the p plane geometry, unless the curve that separates the two dielectrics is determined. We will follow other authors in assuming quasi-TEM propagation. Thus, after determining the combined z to p transformation and the curve between dielectrics, we will be able to calculate the capacitance and the impedance of the line.

A. The z to p Combined Transformation

Equations (1) and (20) that give the z to w and the p to e transformations, respectively, when written in terms of the variable w, are

$$z = -2A\int_0^w \frac{(a' - w^2)\,dw}{\sqrt{(b' - w^2)(1 - w^2)}} \quad (34)$$

$$p = 2B\int_0^w \frac{dw}{\sqrt{(b' - w^2)(1 - w^2)}} = \frac{2B}{\sqrt{b'}}\int_0^w \frac{dw}{\sqrt{\left[1 - w^2(b')^{-1}\right]\left[1 - w^2\right]}}. \quad (35)$$

In view of the definition of the elliptic function sn [10] and the nature of the integral in (35), we can write

$$w = \operatorname{sn}(\boldsymbol{p}) \quad (36)$$

where $\boldsymbol{p}$ is a complex number closely related to p (as we shall shortly see), and

$$\boldsymbol{p} = \int_0^w \frac{dw}{\sqrt{\left[1 - w^2(b')^{-1}\right]\left[1 - w^2\right]}} = F(\phi/m) \quad (37)$$

where

$$\phi = \operatorname{am} \boldsymbol{p} \text{ (the amplitude of } \boldsymbol{p}) \quad (38)$$

$$m = 1/b' \quad (39)$$

so that by combining (35) and (36), we can determine the relationship between p and $\boldsymbol{p}$

$$\boldsymbol{p} = p\sqrt{b'}/(2B). \quad (40)$$

Furthermore, substitution of (36) into (34), use of (9) yields

$$z = \frac{2A}{b'}\left\{\boldsymbol{p}\left[1 - \frac{E'(m)}{K'(m)}\right] - E(\boldsymbol{p}/m)\right\}. \quad (41)$$

Writing Legendre's relation in a modified form yields

$$\frac{E'(m)}{K'(m)} = 1 - \frac{E(m)}{K(m)} + \frac{\pi}{2K(m)K'(m)}. \quad (42)$$

Recalling the form of the complex Jacobi $Z_n(\boldsymbol{p}/m)$ function

$$Z_n(\boldsymbol{p}/m) = E(\boldsymbol{p}/m) - \frac{E(m)}{K(m)}\boldsymbol{p} \quad (43)$$

and using the value of A given by (19), we can write from (41) the combined transformation law from the z to the p plane in two equivalent forms

$$z = \frac{2hK'(m)}{\pi}\left\{Z_n(\boldsymbol{p}/m) + \frac{\pi\boldsymbol{p}}{2K(m)K'(m)}\right\} \quad (44)$$

$$z = \frac{2hK'(m)}{\pi}\left\{Z_n\left(\frac{p\sqrt{b'}}{2B}/m\right) + \frac{\pi}{2K(m)K'(m)}\left(\frac{p\sqrt{b'}}{2B}\right)\right\}. \quad (45)$$

As in the case of one dielectric, in the case of two dielectrics the capacitance will depend on ratios of distances in the p plane. Thus, for our problem, B is arbitrary, and we will select for simplicity $2B = \sqrt{b'}$, so that $p = \boldsymbol{p}$. Through the remainder of the paper we will use (44), with $p = \boldsymbol{p}$.

B. The Curve Between Dielectrics in the p Plane

Fig. 2 shows the nature of the p plane curve; it corresponds to the line $z = h + iy$ that separates the two dielectrics in the z plane (y is a real variable in the range $h \leqslant y \leqslant \infty$). All points u and v on the curve must satisfy (44)

$$h + iy = \frac{2hK'(m)}{\pi}\left\{Z_n[(u + iv)/m] + \frac{\pi(u + iv)}{2K(m)K'(m)}\right\}. \quad (46)$$

Separating the Jacobi Z_n function into its real and imaginary parts yields

$$Z_n[(u+iv)/m] = Z_n(u/m) + \frac{m\,\mathrm{sn}(u/m)\,\mathrm{cn}(u/m)\,\mathrm{dn}(u/m)\,\mathrm{sn}^2(v/m_1)}{1-\mathrm{sn}^2(v/m_1)\,\mathrm{dn}^2(u/m)} - i\left\{ Z_n(v/m_1) + \frac{\pi v}{2K(m)K'(m)} - \frac{\mathrm{dn}^2(u/m)\,\mathrm{cn}(v/m_1)\,\mathrm{sn}(v/m_1)\,\mathrm{dn}(v/m_1)}{1-\mathrm{sn}^2(v/m_1)\,\mathrm{dn}^2(u/m)} \right\} \tag{47}$$

where $m_1 = 1 - m$, and we have introduced the elliptic functions sn, cn, and dn (see [10]). Substituting (47) into (46), and the separate real and imaginary parts to yield

$$y = \frac{2hK'(m)}{\pi}\left\{ -Z_n(v/m_1) + \frac{\mathrm{dn}^2(u/m)\,\mathrm{cn}(v/m_1)\,\mathrm{sn}(v/m_1)\,\mathrm{dn}(v/m_1)}{1-\mathrm{sn}^2(v/m_1)\,\mathrm{dn}^2(u/m)} \right\} \tag{48}$$

and

$$1 = \frac{2K'(m)}{\pi}\left\{ Z_n(u/m) + \frac{\pi u}{2K(m)K'(m)} + \frac{m\,\mathrm{sn}(u/m)\,\mathrm{cn}(u/m)\,\mathrm{dn}(u/m)\,\mathrm{sn}^2(v/m_1)}{1-\mathrm{sn}^2(v/m_1)\,\mathrm{dn}^2(u/m)} \right\}. \tag{49}$$

Equation (49) gives the desired relationship $v = v(u)$ that defines the curve between the dielectrics. In order to see this, let us solve for $\mathrm{sn}(v/m_1)$

$$\mathrm{sn}(v/m_1) \equiv g(u/m_1) = \left\{ \frac{m\,\mathrm{sn}(u/m)\,\mathrm{cn}(u/m)\,\mathrm{dn}(u/m)}{\dfrac{\pi}{2K'(m)} - \dfrac{\pi u}{2K(m)K'(m)} - Z_n(u/m)} + \mathrm{dn}^2(u/m) \right\}^{-1/2} \tag{50}$$

where we have defined the function $g(u/m_1)$ for clarity. From the definition of the inverse sn function we have

$$\mathrm{sn}^{-1}(v/m_1) = v = \int_0^{g(u/m_1)} \frac{dw}{\sqrt{(w^2-1)(m_1 w^2 - 1)}} = F(\phi_3/m_1) \tag{51}$$

where

$$\phi_3 = \sin^{-1}[g(u/m_1)]. \tag{52}$$

We can now resume the procedure which is followed in our computer programs in the evaluation of the curve $v(u)$.

1) Determine m as a function of (d/h) by solving (18).
2) Select a value of u in the range between the plates $(0 \leqslant u \leqslant K)$.
3) Calculate $Z_n(u/m)$, $\mathrm{sn}(u/m)$, and $\mathrm{dn}(u/m)$.
4) Calculate $g(u/m_1)$ from (50).
5) Determine ϕ_3 according to (52).
6) Calculate the desired v as: $v = F(\phi_3/m_1)$.

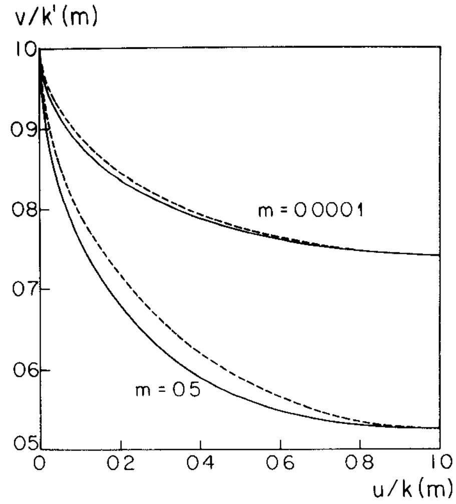

Fig. 4. Curve between dielectrics for two different m values. Continuous lines: exact curves; dotted lines: quarter on an ellipse approximation.

Fig. 4 shows the curve between dielectrics in a normalized diagram ($u' \equiv u/K(m)$ and $V' \equiv v/K'(m)$ for different values of m. Also shown in Fig. 4 are two curves corresponding to quarter of ellipses (for comparison with Wheeler's solutions).

C. The Calculation of the Capacitance for the Two Dielectric Case

Once the curve between dielectrics is known, we proceed to calculate the electrostatic potentials in the two dielectric regions $V_1(u',v')$ and $V_2(u',v')$. Both V_1 and V_2 satisfy the Laplace equation, subject to the following boundary conditions:

$$V_1(0,v') = 0 \tag{53}$$

$$V_1(1,v') = V_2(1,v') = 1 \tag{54}$$

$$\nabla^{\perp} V_1(u',0) = \nabla^{\perp} V_2(u',1) = 0 \tag{55}$$

$$V_1 = V_2 \quad \text{on the curve between} \tag{56}$$

$$\epsilon_1 \nabla^{\perp} V_1 = \epsilon_0 \nabla^{\perp} V_2 \quad \text{dielectrics.} \tag{57}$$

Equation (54) indicates the application of a normalized external voltage, and (57) implies the continuity of the normal component of the electric-field density at the interface between dielectrics. $\nabla^{\perp}$ indicates the normal component of the gradient. Due to the nature of the curve between dielectrics, we solve the problem numerically at discrete voltage nodes, as those indicated schematically in Fig. 5. The details of this calculation are discussed elsewhere [12]. Once the voltages at all nodes in a column next to a metal plate are known, the capacitance C_2 for the two dielectric system can be obtained. In the calculation, we assumed medium 2 to be air ($\epsilon_2 = \epsilon_0$).

Once C_2 is determined, in view of (30), we can define and calculate an effective relative dielectric permittivity ϵ'_{eff}

$$\epsilon^r_{\text{eff}} \equiv C_2 K(m)/[2\epsilon_0 K'(m)]. \tag{58}$$

Thus, for the two dielectric case, when the top one is air, the characteristic line impedance Z_2 (assuming TEM prop-

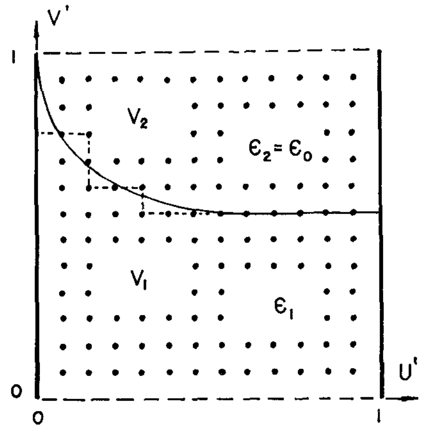

Fig. 5. The geometry of the two dielectrics capacitor. We indicate the position of some of the discrete voltage nodes used in the numerical calculation. The continuous curve separates the dielectrics, and the broken curve indicates its numerical implementation. In our calculation we used a matrix of 100×100 nodes.

TABLE II

d/h	Wheeler Z(Ω)	Poh et al Z(Ω)	Our Results Z(Ω)	ϵ^r_{eff}
0.10	156.900 (-0.02)	156.8150 (-0 07)	156.9334	2.7888
0.20	131.200 (-0 02)	131.0790 (-0.11)	131.2270	2.8362
0 40	105.600 (+0.04)	105.4570 (-0.08)	105.5496	2.9020
0.80	80.360 (+0.60)	80.2888 (+0.51)	79.8765	3.0065
1.00	72 470 (+0.19)	72.4259 (+0.13)	72.3298	3 0553
1.25	64.780 (+0.13)	64.7711 (+0.11)	64.6965	3 1025
1.6666	55.280 (+1.07)	55.3316 (+1.17)	54.6919	3.1789
2.50	43.030 (+0.02)	43.1640 (+0.33)	43.0226	3.2977
5 00	26.270 (-0.18)	26.4120 (+0.35)	26.3188	3.5200
10 00	15.050 (+0.12)	15.0800 (+0.32)	15.0315	3.7304
15 00	10 716 (+1.41)		10.5661	3.8370

Two dielectrics case: ϵ_2 = Free space, $\epsilon_1^r = 4.2$. Our results are compared with those of Wheeler [5] and those of Poh *et al.* [8]. Numbers in parentheses indicate percentage differences with our results.

TABLE III

d/h	Wheeler Z(Ω	Poh et al Z(Ω)	Our Results Z(Ω)	ϵ^r_{eff}
0 10	48.94 (-0 43)	48 8455 (-0.62)	49 1500	28 4321
0.20	40 80 (-0 31)	40.7015 (-0 55)	40 9273	29 1575
0 40	32 70 (-0.11)	32.5941 (-0 43)	32 7362	30 1687
0.80	24.74 (+0 76)	24.6381 (+0.35)	24 5523	31.8215
1 00	22.26 (+0 53)	22.1577 (+0.07)	22 1419	32.6081
1.25	19.84 (+0 55)	19 7478 (+0 08)	19.7317	33.3529
1 6666	16 85 (+1.62)	16.7860 (+1.23)	16 5818	34.5829
2.50	13.03 (+0.74)	12 9930 (+0 46)	12.9340	36.4872
5.00	7.85 (+0 62)	7.8421 (+0 52)	7.8019	40 0576
10.00	4 44 (+0.76)	4 4260 (+0 44)	4.4065	43.4077
15 00	3.14 (+1 87)		3.0821	45 0942

Two dielectrics case: ϵ_2 = free space, $\epsilon_1^r = 51$. Our results are compared with those of Wheeler [5] and those of Poh *et al.* [8] Numbers in parentheses indicate percentage differences with our results

agation) will be given (according to (33)) as

$$Z_2 = \frac{Z_0}{\sqrt{\epsilon^r_{eff}}} \tag{59}$$

where Z_0 is the impedance of the line immersed in air ($\epsilon_1 = \epsilon_2 = \epsilon_0$). Table II shows our impedance results for the case where $\epsilon_1^r = 4.2$, as well as results by Wheeler and Poh *et al.* Table III shows similar results for the case $\epsilon_1^r = 51$.

V. Discussion

Fig. 3 and Table I present the comparison of our results with those of others for the case of one dielectric. It is interesting to compare the finite ground-plane solution of Black and Higgins, and the infinite ground-plane solution. The points shown on Fig. 3, corresponding to Schneider's solution, were calculated using his approximate solution (see [6, eqs. (16) and (17)). One entry in Table I corresponds to the Schneider exact solution evaluated in terms of theta functions, and it agrees exactly with our solution. For the one dielectric case, any of the solutions presented are accurate, with the exception of Assadurian and Rimai. Of greater interest to us is the two dielectric solution. Our results are summarized in Tables II and III. For the case of $\epsilon_1^r = 4.2$ and $\epsilon_0^r = 51$, and for the range of values (d/h) presented, the maximum difference between our results and those of Wheeler, and Poh *et al.*, is of the order of 1 to 2 percent.

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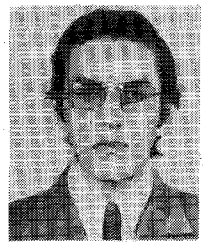

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Analysis of Wave Propagation in Anisotropic Film Waveguides with Bent Optical Axes

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***Abstract*—We present an analytical method for studying the wave propagation in anisotropic planar optical waveguides where the oblique angle between the optical axis and the propagation axis changes arbitrarily in the film surface along the propagation length. The analysis is based on the coupled-mode theory, where the coupling between a guided mode and radiation modes is regarded to be of major importance. We apply a hypothetical boundary method to quantize the continuum of radiation modes, and replace the continuously changing oblique angle by a step approximation. It is shown that these approximations do not degrade the computational accuracy. To exemplify the wave-propagation properties, we deal with a waveguide consisting of $LiNbO_3$ and let the oblique angle change linearly along the propagation length. It is found that the incident guided TE mode leaks its power primarily in a very narrow region centered on the critical oblique angle, and that TE radiation modes play an important role in the power conversion, even though they carry far less power than the TM radiation modes.**

I. Introduction

IT IS OF fundamental interest to know the guiding properties of dielectric optical waveguides composed of anisotropic, as well as isotropic, materials. Such knowledge is needed for applications to guided-wave devices for optical integrated circuits. Usually, two different approaches have been adopted in waveguide analysis. One approach is based on the eigenvalue method in which modal solutions of Maxwell's equations are determined with the help of boundary conditions provided that the waveguide is infinitely long and homogeneous along the propagation axis. Most papers on wave propagation in anisotropic waveguides using this method have dealt with purely guided modes [1]–[6]. Recently, interesting propagation characteristics of hybrid leaky modes supported by planar anisotropic waveguides or metal-diffused anisotropic waveguides have been analyzed where the optical axis of the composing material makes an oblique angle with the propagation axis in the film surface [7], [8].

The other approach is based on the coupled-mode theory [9]. It is suitable for describing the wave propagation in waveguides that are inhomogeneous along the propagation axis and/or of finite length suitable for integrated optics devices. Therefore, propagation properties obtained from it may be useful from the device-planning viewpoint. In the coupled-mode theory, power leakage of a hybrid leaky mode in an anisotropic waveguide is attributed to mode conversion between a guided mode and radiation modes of the orthogonal polarization [10]. The coupled-mode theory is always applicable to the analysis of wave propagation in anisotropic waveguides having any nondiagonal dielectric

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